

# Classical Geometry from a Physical State in Canonical Quantum Gravity

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## Abstract

We construct a weave state which approximates a degenerate 3-metric of rank 2 at large scales. It turns out that a non-degenerate metric region can be evolved from this degenerate metric by the classical Ashtekar equations, hence the degeneracy of 3-metrics is not preserved by the evolution of Ashtekar's equations. As the s-knot state corresponding to this weave is shown to solve all the quantum constraints in loop quantum gravity, a physical state in canonical quantum gravity is related to the familiar classical geometry.

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# 1 Introduction

Since the new Hamiltonian formulation of gravity was proposed by Ashtekar in 1986 [1], considerable progress has been made in non-perturbative canonical quantum gravity, namely, loop quantum gravity [2]. One of the surprising results of the theory is that some solutions of all the quantum constraints in canonical gravity have been found [3, 4]. The kinematic of the theory is now rigorously defined [5, 6]. However, to accept the theory as a conceivable candidate for describing quantum space-time, we need to prove that its classical limit is general relativity (GR) or at least overlaps GR in the regime where GR is well tested.

A weave state was first introduced in Ref.[7] to approximate the flat 3-geometry. Being one of the solutions in Ref.[3], it is regarded as a physical state of loop quantum gravity. However, this state is an eigenstate of the volume operator[8, 9] with vanishing eigenvalue. Moreover, as argued in Refs.[10, 4], the classical correspondences of the solutions in Ref.[3] and their generalisation[4] should all be degenerate metrics which are not admitted in the traditional GR. While, as far as we know, the other weave states appearing so far are all kinematical states at the unconstrained level [11, 12, 13, 14].

As there is growing evidence from various descriptions of quantum gravity that degenerate metrics should have an important role [3, 15, 16], researches have been promoted to study the degenerate metric in the classical Ashtekar theory as it is admitted by the formalism, including the dynamic characters of degenerate triads[17, 18, 19] and degenerate phase boundaries[20, 21, 22]. By breaking the causality, a solution to classical Ashtekar's equations was constructed in Ref.[23], where a degenerate space-time region could be evolved from non-degenerate initial data. However, no example has been raised so far where a non-degenerate metric is generated by the time evolution of degenerate initial data, although it is not impossible in principle.

The present paper involves the above two topics. We will show that, in loop quantum gravity a quantum state based on the s-knot class of an infinite number of open curves can solve all quantum constraints and approximate a degenerate 3-geometry, from which a non-degenerate metric region can be evolved by the classical Ashtekar equations. Thus, a physical state of canonical quantum gravity is related to the familiar classical geometry. In Section 2, we construct a “quasi-coherent” state and show how it can weave a rank 2 degenerate metric on  $R^3$  at large scales. In section 3, we show an example in complex Ashtekar's formalism where a non-degenerate metric is generated by

the time evolution of the degenerate 3-metric. The physical state related to the weave is induced after a few comments and discussions in Section 4.

## 2 Weaving a degenerate metric

### 2.1 Preliminaries

Canonical gravity in the real Ashtekar formalism is defined over an oriented 3-manifold  $\Sigma$  [24]. The basic variables are real  $SU(2)$  connections  $A_a^i$  and densitized triads  $\tilde{E}_j^b$  of weight 1. We use  $a, b, \dots = 1, 2, 3$  for spatial indices and  $i, j, \dots = 1, 2, 3$  for internal  $SU(2)$  indices. A tilde over(under) a letter denotes a density weight 1(-1).

The Ashtekar theory admits a generalisation of GR to involve degenerate metrics, since the inverse of the triads is not necessary for the whole formalism. The triad is related to the 3-metric by

$$\tilde{h}^{ab} = \tilde{E}_i^a \tilde{E}^{bi}. \quad (1)$$

In the case where  $h_{ab}$  is non-degenerate,

$$\left(\tilde{h}^{ab}\right) = h(h^{ab}) = (\mathcal{A}^{ab}), \quad (2)$$

where  $h := \det(h_{ab})$ , and the elements of  $(\mathcal{A}^{ab})$  are the cofactors of  $(h_{ab})$ . Note that Eq.(2) can be naturally generalised to the case where  $h_{ab}$  becomes degenerate by neglecting the middle procedure.

Based on the Ashtekar variables, canonical quantum gravity can also be represented by loop variables, and there exist well-defined non-local operators carrying geometric informations [3, 7]. In the following we will consider the operator  $\hat{Q}[\omega]$  associated with one-forms  $\omega_a$  on  $\Sigma$  [7, 13], rather than the operators of area and volume. One can see,  $\hat{Q}[\omega]$  is more suitable to respect the feature of a degenerate metric, as it is related to the classical quantity

$$Q[\omega] = \int d^3x \sqrt{\tilde{E}_i^a \omega_a \tilde{E}^{bi} \omega_b} = \int d^3x \sqrt{\tilde{h}^{ab} \omega_a \omega_b}, \quad (3)$$

where  $\omega_a$  is any smooth 1-form which makes the integral meaningful and the integral is well defined since the integrand is a density of weight 1. Using loop variables [3, 25], the same quantity could be expressed as

$$Q[\omega] = \lim_{\epsilon \rightarrow 0} \int d^3x \left[ \int d^3y \int d^3z f_\epsilon(x, y) f_\epsilon(x, z) \frac{1}{2} T^{ab}[\alpha_{yz}](y, z) \omega_a(y) \omega_b(z) \right]^{\frac{1}{2}}, \quad (4)$$

where  $f_\epsilon(x, y)$  tends to  $\delta(x, y)$  as  $\epsilon$  tends to zero,  $\alpha_{yz}$  is an arbitrarily defined smooth loop in  $\Sigma$  that passes through points  $y$  and  $z$  such that it goes to a point as  $y \rightarrow z$ , and the loop variable

$$\mathcal{T}^{ab}[\alpha](y, z) = -Tr[\rho_1(H_\alpha(y, z)\tilde{E}^b(z)H_\alpha(z, y)\tilde{E}^a(y))], \quad (5)$$

here  $H_\alpha(y, z) := \mathcal{P}exp[-\int_z^y ds \dot{\alpha}^a A_a(\alpha(s))]$  is the holonomy or parallel propagator of the connection along  $\alpha$ , and the 2-dimensional representation,  $\rho_1$ , of  $SU(2)$  is used to evaluate traces. Eq.(4) is valid for the quantum version: One can get the well-defined quantum operator  $\hat{Q}[\omega]$  simply by replacing  $\mathcal{T}^{ab}$  by the loop operator  $\hat{\mathcal{T}}^{ab}$  [7, 25]. The action of this operator on a coloured loop state gives[13]

$$\hat{Q}[\omega]\Psi_P[\gamma] = 16\pi l_{Pl}^2 \sqrt{\frac{P}{2}(\frac{P}{2} + 1)} \int_\gamma ds |\dot{\gamma}^a \omega_a(\gamma(s))| \Psi_P[\gamma], \quad (6)$$

where  $P$  is the positive integer associated with the loop  $\gamma$ ,  $l_{Pl}$  denotes the Planck length, and  $\Psi_P[\gamma] := Tr[\rho_P(H[\gamma])]$ , here  $\rho_P$  denotes the  $(P+1)$ -dimensional representation of  $SU(2)$ .

We now briefly introduce the Hilbert space of loop quantum gravity [5, 6]. Given any graph,  $\Gamma_n = \{\gamma_1, \dots, \gamma_n\}$ , embedded in  $\Sigma$  and a function  $f_n : [SU(2)]^n \rightarrow \mathbb{C}$ , the cylindrical function is defined as:

$$\Psi_{\Gamma_n, f_n}(A) := f_n(H[\gamma_1], \dots, H[\gamma_n]). \quad (7)$$

Since any two cylindrical functions based on different graphs can always be viewed as being defined on the same graph which is just constructed as the union of the original ones, it is straightforward to define a scalar product for them by:

$$\langle \Psi_{\Gamma_n, f_n} | \Psi_{\Gamma_n, g_n} \rangle := \int_{[SU(2)]^n} dH_1 \dots dH_n \overline{f_n(H_1, \dots, H_n)} g_n(H_1, \dots, H_n), \quad (8)$$

where  $dH \dots dH_n$  is the Haar measure of  $[SU(2)]^n$  which is naturally induced by that of  $SU(2)$ . The Hilbert space,  $\mathcal{H}$ , is obtained by completing the space of all finite linear combinations of cylindrical functions in the norm induced by the quadratic form (8) on a cylindrical function.

The operators of area and volume have been shown to be self-adjoint on  $\mathcal{H}$  [26, 9]. Since  $\hat{Q}[\omega]$  is closely related to the area operator [13], it is reasonable to conceive it is also self-adjoint. This conceit can be rigorously proved [27].

## 2.2 The weave

The geometry which we want to approximate is a degenerate "flat" 3-metric,  $h_{ab}$ , of rank 2 on  $R^3$ . The metric is "flat" in the sense that there exists a foliation  $R^3 = R^2 \times R$  such that the induced

2-metric,  $q_{ab}$ , of  $h_{ab}$  on  $R^2$  is the flat Euclidean metric.

Let  $\{X, Y, Z\}$  be the Cartesian coordinates on  $R^3$  compatible with the decomposition  $R^3 = R^2 \times R$  and  $(\frac{\partial}{\partial Z})^a$  be the degenerate vector field of  $h_{ab}$ . Thus the line element of  $h_{ab}$  reads

$$ds^2 = dX^2 + dY^2. \quad (9)$$

Hence, from Eq.(2) the only non-zero component of  $\tilde{h}^{ab}$  is  $\tilde{h}^{ZZ} = 1$ .

The weave states which approximate classical 3-metrics were first constructed as the eigenstates of geometrical operators such as  $\hat{Q}[\omega]$  and the operators of area and volume [7, 11, 13]. The corresponding eigenvalues are required to agree with the classical values of the geometrical quantities at large scales. The updated successful construction of  $\mathcal{H}$  promotes us now to approximate a classical geometry by the expectation values of the geometrical operators.

For the operator  $\hat{Q}[\omega]$ , one can define the following: A quantum state  $\Psi$  is said to approximate a classical metric on  $\Sigma$  at scales larger than a macroscopic length scale  $L$  accessible by current measurement if, for all  $\omega_a$  on  $\Sigma$ ,

$$(i) \quad \langle Q \rangle := \langle \Psi | \hat{Q}[\omega] | \Psi \rangle = Q[\omega] + O\left(\frac{\delta}{L}\right). \quad (10)$$

$$(ii) \quad \Delta_Q := (\langle Q^2 \rangle - \langle Q \rangle^2)^{\frac{1}{2}} \ll Q[\omega]. \quad (11)$$

where  $\delta$  is a fixed length chosen as  $l_{Pl} < \delta \ll L$ . However, this definition may face obstruction when it is used for non-compact  $\Sigma$ , such as  $R^3$ . As argued in Ref.[14], the weave states which describe the geometries on  $R^3$  have to be based on graphs of an infinite number of curves, while the states in  $\mathcal{H}$  constructed so far are based on graphs of finite collections of curves. We now think of a way to overcome the obstruction to a certain extent.

Suppose there is a cover  $\{C_i\}$ , consisted of 3-dimensional regions  $C_i$ , of a non-compact  $\Sigma$ , such that for any  $C_i$ , a weave state  $W_\Sigma$  based on a graph  $\Gamma$  (may consist of an infinite number of curves) can always be expressed as:

$$W_\Sigma = W_{C_i} W_{\Sigma - C_i}, \quad (12)$$

where the cylindrical functions  $W_{C_i}$  and  $W_{\Sigma - C_i}$  are based, respectively, on the subgraphs of  $\Gamma$  restricted to  $C_i$  and  $\Sigma - C_i$ , and the subgraphs of the regions  $C_i$  all consist of finite numbers of curves. Then we can define that  $W_\Sigma$  approximates a classical metric on  $\Sigma$  if all  $W_{C_i}$  approximate, according to Eqs. (10) and (11), the metrics restricted to  $C_i$ . Note that this definition is valid for all of the weaves states and their 3-metrics appeared so far.

We now construct a weave state which approximate the above given degenerate metric  $h_{ab}$ . The basic idea is to consider a family of an infinite number of non-intersecting open curves,  $\{\gamma_i\}$ , instead of closed loops on  $R^3$ . All of the  $\gamma_i$  are required to be the integral curves of the degenerate vector field of  $h_{ab}$ , and hence match the  $Z$ -coordinate curves exactly. This kind of curve was called “large loops” in Refs.[11, 12]. Using the induced 2-metric  $q_{ab}$  on a 2-surface  $Z = \text{const.}$ , we fix the intersections of  $\gamma_i$  and the surface as the lattice sites of a square lattice on  $R^2$  with lattice spacing  $\lambda$ . As mentioned in Ref.[14], a way of dealing with states based on curves of infinite length is to consider a compactification of  $\Sigma$  [28]. Thus  $\gamma_i$  may also be regarded as a closed loop on  $\bar{R}^3$ , where  $\bar{R}^3 := R^3 \cup \infty$  is the one-point compactification of  $R^3$ . Follow Ref.[14], we define the desired “quasi-coherent” state,  $W_{\{\}}$ , based on  $\{\gamma_i\}$  as:

$$W_{\{\}} := \lim_{n \rightarrow \infty} \prod_{i=1}^n \psi_i \quad (13)$$

where

$$\psi_i := \eta \exp(\beta \text{Tr}[\rho_1(H[\gamma_i] - e)]) , \quad (14)$$

here,  $\beta$  is an arbitrary constant,  $e$  is the identity in  $SU(2)$ , and  $\eta$  is a normalisation factor.

To see if  $W_{\{\}}$  weaves the classical geometry determined by  $h_{ab}$ , let us consider a cover  $\{\mathcal{O}_m\}$  of  $R^3$ , where  $\mathcal{O}_m$  denotes the 3-dimensional region  $\{(X, Y, Z) | X^2 + Y^2 < m^2, m \in N\}$ , here,  $N$  is the collection of nature numbers. Let  $n$  be the number of curves  $\gamma_i$  in region  $\mathcal{O}_m$ , it is obvious from Eqs. (13) and (14) that, for any  $\mathcal{O}_m$ ,

$$W_{\{\}} = W_n W_{\{-n\}}, \quad (15)$$

where  $W_n$  and  $W_{\{-n\}}$  are based, respectively, on the graphs  $\{\gamma_i \subset \mathcal{O}_m\}$  and  $\{\gamma_j \subset (R^3 - \mathcal{O}_m)\}$ , which are the subgraphs of  $\{\gamma_i\}$  restricted, respectively, to  $\mathcal{O}_m$  and  $(R^3 - \mathcal{O}_m)$ , and

$$W_n = \prod_{i=1}^n \psi_i. \quad (16)$$

The remaining task is to prove that  $W_n$  approximates the geometry of  $h_{ab}$  on  $\mathcal{O}_m$ . Calculations similar to that of Ref.[14] show that  $\psi_i$  can be expanded in terms of the eigenstates of  $\hat{Q}[\omega]$  as:

$$\psi_i = \sum_{P=0}^{\infty} s_P \Psi_P[\gamma_i], \quad (17)$$

where

$$s_P = \frac{I_P(2\beta) - I_{P+2}(2\beta)}{\sqrt{I_0(4\beta) - I_2(4\beta)}}, \quad (18)$$

here,  $I_P(x)$  is the modified Bessel function of order  $P$ .

From Eqs. (6), (16), and (17) we obtain the expectation value of  $\hat{Q}[\omega]$ ,

$$\begin{aligned}\langle W_n | \hat{Q}[\omega] | W_n \rangle &= 16\pi l_{Pl}^2 \sum_{\gamma_i, i=1}^n \sum_{P=0}^{\infty} s_P^2 \sqrt{\frac{P}{2}(\frac{P}{2}+1)} \int_{\gamma_i} |\omega_Z| dZ \\ &= 16\pi l_{Pl}^2 \sum_{P=0}^{\infty} s_P^2 \sqrt{\frac{P}{2}(\frac{P}{2}+1)} \left( \frac{1}{\lambda^2} \right) \int_{\mathcal{O}_m} |\omega_Z| dX dY dZ + O\left(\frac{\lambda}{L}\right).\end{aligned}\quad (19)$$

Let

$$\lambda = l_{Pl} \left[ 16\pi \sum_{P=0}^{\infty} s_P^2 \sqrt{\frac{P}{2}(\frac{P}{2}+1)} \right]^{\frac{1}{2}}, \quad (20)$$

then, from Eqs. (3) and (19) we have, on region  $\mathcal{O}_m$ ,

$$\langle W_n | \hat{Q}[\omega] | W_n \rangle = Q[\omega] + O\left(\frac{\lambda}{L}\right). \quad (21)$$

Furthermore, straitforward calculations yield

$$[\langle W_n | \hat{Q}^2 | W_n \rangle - (\langle W_n | \hat{Q} | W_n \rangle)^2]^{\frac{1}{2}} = l_{Pl} \xi \left[ \int_{X^2+Y^2 < m^2} dX dY \left( \int |\omega_Z| dZ \right)^2 + O\left(\frac{\lambda}{L}\right) \right]^{\frac{1}{2}}, \quad (22)$$

where

$$\xi = \left[ \frac{\sum_{P=0}^{\infty} s_P^2 \frac{P}{2}(\frac{P}{2}+1) - \left( \sum_{P=0}^{\infty} s_P^2 \sqrt{\frac{P}{2}(\frac{P}{2}+1)} \right)^2}{\sum_{P=0}^{\infty} s_P^2 \sqrt{\frac{P}{2}(\frac{P}{2}+1)}} \right]^{\frac{1}{2}}. \quad (23)$$

Taking account of  $\int dX dY (\int |\omega_Z| dZ)^2 \sim (\int dX dY \int |\omega_Z| dZ)^2$  and the order of  $\xi$ , Eq.(22) is estimated as:

$$[\langle W_n | \hat{Q}^2 | W_n \rangle - (\langle W_n | \hat{Q} | W_n \rangle)^2]^{\frac{1}{2}} \sim l_{Pl} Q[\omega] \ll Q[\omega]. \quad (24)$$

We conclude from Eqs. (22) and (24) that  $W_{\{\}}$  approximates the degenerate metric  $h_{ab}$  on  $R^3$  at scales larger than  $L$ .

The concrete values of  $\lambda$  and  $\xi$  can be obtained from Eqs. (20) and (23) by fixing a particular value of  $\beta$ . For example, we have

$$\beta = 20 : \quad \lambda = \sqrt{3.545(16\pi l_{Pl}^2)} = 13.35 l_{Pl}, \quad \xi = \sqrt{\frac{16\pi 1.509}{3.545}} = 4.63; \quad (25)$$

$$\beta = 40 : \quad \lambda = \sqrt{5.695(16\pi l_{Pl}^2)} = 16.92 l_{Pl}, \quad \xi = \sqrt{\frac{16\pi 1.164}{5.695}} = 3.21. \quad (26)$$

The "quasi-coherent" feature of the weave  $W_{\{\}} can be seen from its construction of Eqs. (13), (14), and (16). The functions  $W_n$  take on their maximum values when  $H[\gamma_i] = e$  and hence, as  $n \rightarrow \infty$ , the function  $W_n$  becomes increasingly peaked around the connections  $A_a^i$  which give a trivial holonomy along all curves  $\gamma_i$ .$

Note that the functions  $Q[\omega]$  carry sufficient information about the 3-metric. If we know  $Q[\omega]$  for every smooth 1-form  $\omega_a$ , the metric is known completely. Using the area operator [7, 26], it is not difficult to check that the weave  $W_{\{\}}$  will reproduce as well the correct values of the areas of any 2-surfaces measured by  $h_{ab}$  in  $R^3$ . Since the curves  $\gamma_i$  are non-intersecting,  $W_{\{\}}$  will give a zero expectation value of the volume operator[25, 9] for any 3-dimensional regions. This is the right result because  $h_{ab}$  is degenerate.

### 3 Evolving a non-degenerate metric from the degenerate one

We will show in this section that a non-degenerate space-time region can be evolved by the classical Ashtekar equations from the degenerate 3-metric woven in last section. We would like to use the complex Ashtekar formalism [1], though the real Ashtekar formalism is preferred in studying quantum states such as that in the last section. Now both of the basic variables  $A_a^i$  and  $\tilde{E}_i^a$  are complexified. The constraint and evolution equations take rather simply forms as follows [1]:

$$\mathcal{D}_a \tilde{E}_i^a = 0, \quad \tilde{E}_i^a F_{ab}^i = 0, \quad \tilde{E}_i^a \tilde{E}_j^b F_{abj} \epsilon^{ijk} = 0, \quad (27)$$

$$\dot{A}_b^i = i \underline{N} \tilde{E}_j^a F_{abk} \epsilon^{ijk} + N^a F_{ab}^i, \quad (28)$$

$$\dot{\tilde{E}}_i^b = -i \mathcal{D}_a (\underline{N} \tilde{E}^{aj} \tilde{E}^{bk}) \epsilon_{ijk} + 2 \mathcal{D}_a (N^{[a} \tilde{E}_i^{b]}), \quad (29)$$

where  $\mathcal{D}_a$  and  $F_{ab}^i$  are, respectively, the derivative operator and curvature associated with  $A_a^i$ , and  $\underline{N}$  and  $N^a$  are, respectively, the lapse density (weight -1) and the shift vector. To recover a real theory, the reality condition that the metric constructed from  $\tilde{E}_i^a$  by Eq.(1) and its time derivative should be real has to be posed.

We now construct the desired example by applying some reparametrization procedure[20, 21] to the Minkowski metric. Consider the Minkowski line element in double-null coordinates  $\{U, V, X, Y\}$ :

$$ds^2 = -dUdV + dX^2 + dY^2. \quad (30)$$



Under the reparametrizations  $U = U(u)$  and  $V = V(v)$ , it becomes

$$ds^2 = -U'V'dudv + dX^2 + dY^2, \quad (31)$$

where  $U' := dU/du$  and  $V' := dV/dv$ . In order to get the desired solution, we define the functions  $U$  and  $V$  as follows:

$$U(x) = V(x) := \begin{cases} x^r, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases} \quad (32)$$

where the real number  $r \geq 3$ . A simple coordinate transformation

$$u = t - Z, \quad v = t + Z, \quad (33)$$

turns the metric (31) into

$$ds^2 = U'(t - Z)V'(t + Z)(-dt^2 + dZ^2) + dX^2 + dY^2. \quad (34)$$

Consider this metric on  $R^4$  covered by coordinates  $\{t, X, Y, Z\}$ , Eq.(32) means it is non-degenerate in the wedge region  $\{u > 0\} \cap \{v > 0\}$  and degenerate outside. The key point is that the space-time with metric (34) represents an evolution of some conjugate pair  $(A_a^i, \tilde{E}_i^a)$  on  $R^3$  covered by  $\{X, Y, Z\}$ , satisfying the Ashtekar equations (27), (28), and (29) as well as the reality condition, and hence is a solution of the Ashtekar theory. This conjugate pair reads,

$$(A_a^i) = 0 \quad (35)$$

and

$$(\tilde{E}_i^a) = \begin{pmatrix} \frac{1}{2}(U' + V') & \frac{i}{2}(U' - V') & 0 \\ -\frac{i}{2}(U' - V') & \frac{1}{2}(U' + V') & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (36)$$

where the rows of  $(\tilde{E}_i^a)$  are  $X, Y, Z$  components, with lapse density and shift vector

$$\underline{N} = 1, \quad N^a = 0. \quad (37)$$

It is straightforward to check that Eqs. (35), (36), and (37) indeed satisfy Eqs. (27), (28), and (29), and the space-time metric constructed from them is the same as Eq.(34). The interesting feature of this solution is that the 3-metric on  $R^3$ , being degenerate initially, becomes partially non-degenerate in the time evolution as can be seen in the space-time diagram (Figure 1).

The phase boundaries between the degenerate and non-degenerate regions are null hypersurfaces agreeing with the conclusion of Ref.[22]. Moreover, before the non-degenerate metric appears, i.e.,

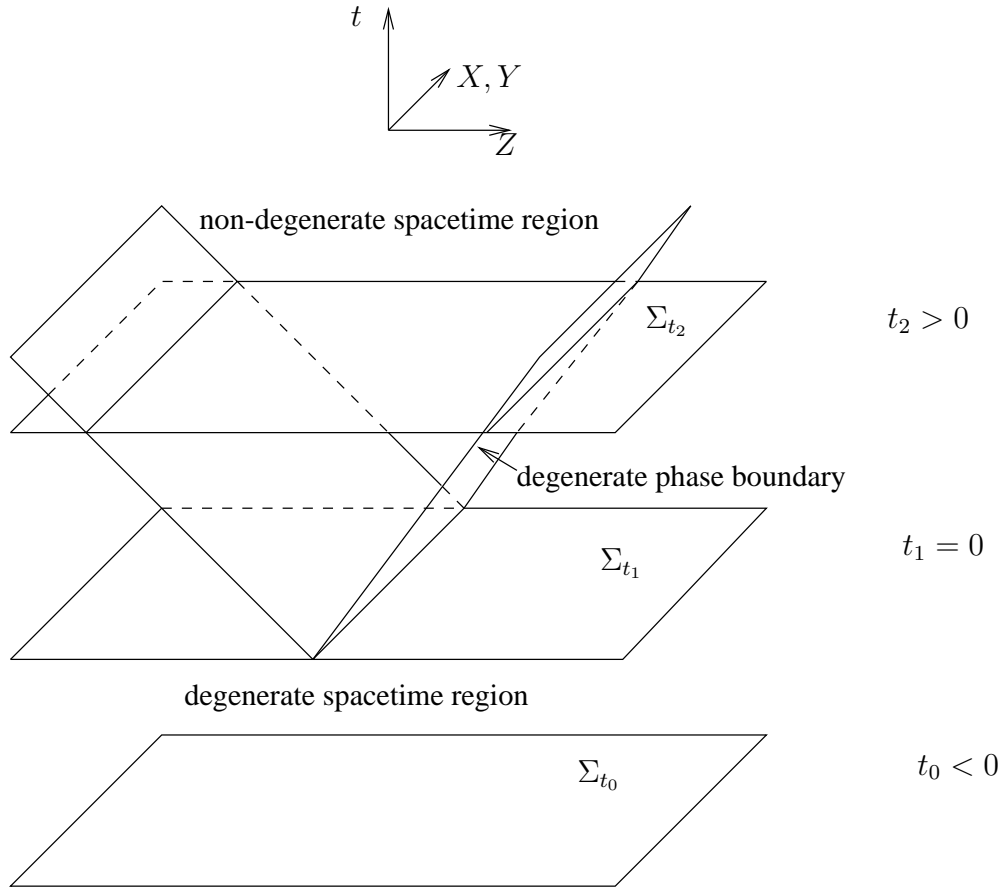


Figure 1: An evolution of a non-degenerate space-time region from the degenerate initial data

for any  $t \leq 0$ , the induced spatial metric of Eq.(34) is of rank 2 and exactly the same as metric (9) on  $R^3$ , which is approximated by the weave state  $W_{\{\}}.$  In other words,  $W_{\{\}}$  has approximated the degenerate 3-metric from which a non-degenerate metric region can be evolved by the classical Ashtekar equations. Another approving point is that the  $SU(2)$  connection (35) gives a trivial holonomy along all curves on  $R^3$ , and hence is one of the connections peaked around by the same weave state.

Note that metric (34) is not  $C^\infty$  at surfaces  $u = 0$  and  $v = 0$ . However, one can let the power  $r$  in Eq.(32) be large enough to obtain some desired differentiability. Note also that we could choose other initial data of  $(A_a^i, \tilde{E}_i^a)$  in the gauge giving the same  $h_{ab}$ , from which a degenerate metric on the whole space- time would be evolved. This supports the observation in Ref.[16] that there are gauge transformations which relate degenerate and non-degenerate metrics.

## 4 Comments and discussion

The weave state based on “large loops” was first proposed in Ref.[29] to approximate a flat 3-metric. But, further investigations show that this kind of weave can not give the “correct eigenvalue” for the area operator [11, 12]. However, our construction shows that “large loops” are well suited to weave degenerate metrics without any problem for the area operator, because the presence of preferred directions of the curves just respects the feature of degenerate metrics.

Our example in Section 3 shows that the degeneracy of 3-metrics is not preserved by Ashtekar’s equations, although it is concluded in Ref.[18] that the “degeneracy type of triads” is locally preserved by the evolution. Moreover, in contrast to the solution in Ref.[23] where the causality has to be broken in order to evolve a degenerate metric from non-degenerate initial data, the causal structure, which may be degenerate [30], of the whole space-time can be still well without any breaking in the inverse evolution. In this sense the non-degenerate region in the example is causally evolved from the degenerate initial data.

It is straitforward to see that the weave state  $W_{\{\}}$  solves the quantum Hamiltonian constraint. A common point to all different regularisation procedures in loop quantum gravity is that the Hamiltonian constraint operator acts only on the nodes of spin networks [31, 4]. From the definition (13) and Eq.(17), it is obvious that  $W_{\{\}}$  can be expanded by spin network basis as:

$$W_{\{\}} = \sum_{\{P_i\}} c_{\{P_i\}} \Psi_{\{P_i\}}, \quad (38)$$

where,  $\Psi_{\{P_i\}} = \lim_{n \rightarrow \infty} \prod_{i=1}^n \Psi_{P_i}[\gamma_i]$  is based on the spin network  $\{P_i\}$  which is obtained by colouring  $P_i$  to every  $\gamma_i$ , and the sum is over all possible choices of colouring of  $\gamma_i$ . Since the graph  $\{\gamma_i\}$  consists of non-intersecting curves,  $\Psi_{\{P_i\}}$  and hence  $W_{\{\}}$  are annihilated by the Hamiltonian constraint operator. In fact,  $W_{\{\}}$  can be viewed as a special kind of combinatorial solution in Ref.[4].

To get the state solving the diffeomorphism constraint, we use the loop representation[3, 32] and define the spin network state  $\Phi_{K\{P_i\}}$  on non-intersecting coloured curves  $\alpha_{p'}$  by:

$$\Phi_{K\{P_i\}}[\alpha_{p'}] := \begin{cases} 1, & \text{if } \alpha_{p'} \in K(\{P_i\}) \\ 0, & \text{otherwise} \end{cases} \quad (39)$$

where the s-knot  $K(\{P_i\})$  is the equivalence class of the embedded spin networks  $\{P_i\}$  under the action of the diffeomorphism group,  $Diff(R^3)$ , on  $R^3$ , i.e.,  $\{P_i\}, \{P'_j\} \in K$ , if there exists a  $\phi \in Diff(R^3)$ , such that  $\{P'_j\} = \phi \cdot \{P_i\}$ . Replacing the spin network basis  $\Psi_{\{P_i\}}$  in Eq.(38) by the diffeomorphism-invariant knot states  $\Phi_{K\{P_i\}}$ , we obtain the corresponding quantum state:

$$\mathcal{W}_{K\{\}} = \sum_{\{P_i\}} c_{\{P_i\}} \Phi_{K\{P_i\}}, \quad (40)$$

which solves all the quantum constraints. Hence,  $\mathcal{W}_{K\{\}}$  should be a physical state of loop quantum gravity for  $R^3$ , although it does not belong to the Hilbert space constructed currently for the states based on graphs of a finite number of curves. Since the spin network  $\{P_i\}$  corresponds to a rank 2 degenerate flat metric  $h_{ab}$ , the equivalence class  $K(\{P_i\})$  of spin networks should correspond to the equivalence class of all metrics related to  $h_{ab}$  by a spatial diffeomorphism. Thus it is natural to interpret  $\mathcal{W}_{K\{\}}$  as representing the rank 2 degenerate flat 3-geometry at large scales.

Moreover, the result in Section 3 shows that this degenerate 3-geometry can be related to some locally non-degenerate geometry by classical Ashtekar's equations, and hence it plays the role of a bridge between a physical state in canonical quantum gravity and the familiar classical geometry.

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